Nonequilibrium Field Theories and Stochastic Dynamics

Zehong Liao

December 15, 2025

1 introduction

In previous courses, we have learned two primary methods for describing stochastic processes: **Langevin Equations** and **Fokker-Planck Equations**. The Langevin Equation starts from a microscopic perspective, tracking the specific trajectory of a single particle under the combined action of deterministic and random forces; whereas the Fokker-Planck Equation adopts a macroscopic statistical perspective, describing how the probability density of an ensemble composed of a large number of particles evolves over time. These two methods correspond to tracking the growth of "a single tree" versus depicting the changes of a "whole forest," respectively.

This lesson will introduce a brand-new, powerful theoretical framework—**Path Integral**. This method provides us with a third perspective on stochastic dynamics. It is no longer limited to describing the evolution of the system state from one instant to the next, but rather focuses on the complete history or trajectory $x(\tau)$ over the entire time interval. The core idea of the path integral is to assign a probability weight to every complete path the system might experience. By "summing" over all possible paths (i.e., **Functional Integration**), we can calculate various physical quantities of interest, such as the total probability of transitioning from one state to another.

This concept is strikingly similar to the Feynman path integral in quantum mechanics. In quantum mechanics, the transition amplitude of a particle from point A to point B is obtained by integrating the phase factor $e^{iS/\hbar}$ over all possible paths connecting A and B, where S is the classical action. In stochastic dynamics, we will adopt a similar approach, but the weight of the integration is no longer a complex phase factor, but a real-valued probability, usually in the form of e^{-S}, where S is a functional we call "**Action**", which measures the "improbability" or "cost" of a specific path occurring.

By constructing this framework, we will be able to answer not only questions like "Where will the particle appear at the next moment?" but also profound questions like "Which path is the particle most likely to follow to get from point A to point B within a certain period?" This perspective is not only theoretically elegant but also demonstrates strong practical value in many frontier fields of modern physics, such as soft matter physics, biophysics, and financial modeling.

2 Path integral of a stochastic process

We discretize the time interval $[0, t]$ into n small time steps Δt, such that $t = n\Delta t$. At each time point $\tau_i = i\Delta t$ (where $i = 0, 1, \ldots, n-1$), the value of the noise is $\xi_i = \xi(\tau_i)$. Since we are dealing with white noise, the noise at different time points is statistically independent. Therefore, we can regard each ξ_i as an independent, Gaussian-distributed random variable. Its probability density function is given by:

$$p(\xi_i) = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left[-\frac{\xi_i^2}{2\sigma_\xi^2}\right] \tag{1}$$

where σ_ξ^2 is the variance of the noise within a single time step Δt.

Since each ξ_i is **independently and identically distributed (i.i.d.)**, the joint probability of observing a specific discrete noise path $(\xi_0, \xi_1, \ldots, \xi_{n-1})$ is the product of all individual probabilities:

$$p(\xi_0, \xi_1, \ldots, \xi_{n-1}) = \prod_{i=0}^{n-1} p(\xi_i) = \left(\frac{1}{2\pi\sigma_\xi^2}\right)^{n/2} \exp\left(-\sum_{i=0}^{n-1} \frac{\xi_i^2}{2\sigma_\xi^2}\right) \tag{2}$$

We are mainly concerned with the exponential part, as it determines the relative magnitude of the probability.

Now, we perform a crucial step: letting the time step $\Delta t \to 0$ and the number of steps $n \to \infty$, while keeping the total time $t = n\Delta t$ constant. In this limit, the discrete summation will transition to a continuous integral.

First, we need to establish the relationship between the discrete variance σ_ξ^2 and the continuous diffusion coefficient D. We achieve this by matching the noise correlation functions in the discrete and continuous forms.

In the discrete form, the correlation is given by

$$\langle \xi_i \xi_j \rangle = \sigma_\xi^2 \delta_{ij}.$$

In the continuous form, the correlation is given by

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t').$$

Averaging the continuous form over the discrete time points yields:

$$\begin{aligned}
\langle \xi_i \xi_j \rangle &\approx \frac{1}{(\Delta t)^2} \int_{\tau_i}^{\tau_{i+1}} d\tau \int_{\tau_j}^{\tau_{j+1}} d\tau' \langle \xi(\tau)\xi(\tau') \rangle \\
&= \frac{1}{(\Delta t)^2} \int_{\tau_i}^{\tau_{i+1}} d\tau \int_{\tau_j}^{\tau_{j+1}} d\tau' 2D\delta(\tau - \tau') \\
&= \frac{2D}{(\Delta t)^2} \Delta t \delta_{ij} \\
&= \frac{2D}{\Delta t} \delta_{ij}
\end{aligned}$$

Comparing the two forms, we obtain the relation $\sigma_\xi^2 = 2D/\Delta t$.

Now, we substitute this relation into the exponent of the joint probability:

$$\begin{aligned}
-\sum_{i=0}^{n-1} \frac{\xi_i^2}{2\sigma_\xi^2} &= -\sum_{i=0}^{n-1} \frac{\xi_i^2}{2(2D/\Delta t)} \\
&= -\frac{1}{4D} \sum_{i=0}^{n-1} \xi_i^2 \Delta t
\end{aligned}$$

As $\Delta t \to 0$, the sum (Riemann sum) transforms into a continuous integral:

$$\sum_{i=0}^{n-1} \xi_i^2 \Delta t \xrightarrow{\Delta t \to 0} \int_0^t d\tau\, \xi^2(\tau) \tag{3}$$

Formally, the complete path integral is written as:

$$\langle O[\xi(\tau)] \rangle = \int \mathcal{D}[\xi(\tau)]\, O[\xi(\tau)] \exp\left(-\frac{1}{4D} \int_0^t d\tau\, \xi^2(\tau) \right) \tag{4}$$

where $O[\xi(\tau)]$ is an arbitrary observable (functional) that depends on the noise path. $\mathcal{D}[\xi(\tau)]$ is the so-called "**path integral measure**", which represents the summation over all possible paths and can formally be understood as $\lim_{n\to\infty} \prod_{i=0}^{n-1} \frac{d\xi_i}{\sqrt{2\pi\sigma_\xi^2}}$, containing the normalization factor.

3 From Noise to Particles: Onsag-Machlupp Action

We have successfully constructed the probability functional for the noise path, $P[\xi(\tau)]$. Now, our core task is to use it to find the probability functional for the particle path, $P[x(\tau)]$. The key strategy here is to perform a **change of variables** in the functional integral, transforming the integration from the noise path $\xi(\tau)$ to the particle path $x(\tau)$.

In the functional integral, when transforming from the noise path $\xi(\tau)$ to the particle path $x(\tau)$, the relationship between the two paths is given by the Langevin equation:

$$\dot{x}(\tau) = A(x(\tau)) + \xi(\tau).$$

Rearranging this provides the explicit relationship for the noise path:

$$\xi(\tau) = \dot{x}(\tau) - A(x(\tau)).$$

The transformation from the integration measure $\mathcal{D}[\xi(\tau)]$ to $\mathcal{D}[x(\tau)]$ requires multiplying by the Jacobian determinant J of the transformation, which is the functional derivative of $\xi(\tau)$ with respect to $x(\tau)$:

$$J = \mathrm{Det}\left(\frac{\delta\xi(\tau)}{\delta x(\tau')}\right).$$

For the Langevin equation $\dot{x} = A(x) + \xi$, since ξ is a function of $\dot{x}$ and x, the Jacobian determinant in this case is a non-trivial constant. Its value, which is independent of the path $x(\tau)$, is:

$$J = \prod_{i=0}^{n-1} \left|\frac{\partial \xi_i}{\partial x_i}\right| \tag{5}$$

For the specific form of the Langevin equation $\dot{x} = A(x) + \xi$, the Jacobian determinant J is given by:

$$J = \exp\left(-\frac{1}{2}\int_0^t d\tau\, \frac{\partial A(x)}{\partial x}\right) \tag{6}$$

This means that the full probability functional for the particle path $x(\tau)$ is:

$$P[x(\tau)] \sim J \cdot P[\xi(\tau)] \sim \exp\left(-\frac{1}{4D}\int_0^t d\tau\, (\dot{x} - A(x))^2 - \frac{1}{2}\int_0^t d\tau\, \frac{\partial A(x)}{\partial x}\right) \tag{7}$$

In understanding the functional determinant, which is a key part of the transformation, and to trace its origin, we return to the discrete picture.

The discrete form of the Langevin equation can be written as:

$$x_{i+1} - x_i = \Delta t\,[\theta A(x_{i+1}) + (1-\theta)A(x_i)] + \sqrt{2D\Delta t}\xi_i \tag{8}$$

where $\theta \in [0, 1]$ is a parameter that represents the time point chosen for evaluating the drift term $A(x)$.

- $\theta = 0$ corresponds to the **Itô interpretation**, using the position at the start of the time step, x_i.
- $\theta = 1$ corresponds to the **anti-Itô interpretation** (or terminal point), using the position at the end of the time step, x_{i+1}.
- $\theta = 1/2$ corresponds to the **Stratonovich interpretation**, using the midpoint approximation.

The solutions of stochastic differential equations (SDEs) under the Itô, Stratonovich, and anti-Itô interpretations are different because the value of the drift function $A(x)$ is evaluated at different points within the interval $[t_i, t_{i+1}]$. In other words, the choice of $\theta \in [0, 1]$ controls the interpretation: $\theta = 0$ (Itô) corresponds to evaluating $A(x)$ at the beginning of the step; $\theta = 1/2$ (Stratonovich) corresponds to the midpoint approximation, which preserves the classical chain rule and is often used in physical systems near the thermodynamic limit; $\theta = 1$ (anti-Itô) corresponds to evaluating $A(x)$ at the end of the step. This difference in discretization method affects not only the numerical stability but, more importantly, determines the Jacobian correction term in the path integral, leading to different continuum limits and physical/mathematical consequences.

In the language of the path integral, the choice of interpretation corresponds to deciding at which point in the path the particle "chooses" the direction of its next step.

- Itô only considers the current direction (safest for finance).

- Stratonovich estimates the direction at the intermediate time (more physically reasonable, e.g., in mechanics).
- Anti-Itô assumes "future knowledge" and looks at the final direction (unphysical in reality).

This choice represents the different conventions we adopt to deal with the ambiguity of "how the sampling rule" is chosen in the discrete process, and whether we want the resulting quantity to be mathematically simple or physically realistic.

From the recurrence relation, we can solve for the noise term ξ_i:

$$\xi_i = \frac{x_{i+1} - x_i}{\sqrt{2D\Delta t}} - \frac{\Delta t}{\sqrt{2D\Delta t}} \left[\theta A(x_{i+1}) + (1-\theta)A(x_i)\right] \tag{9}$$

The element of the Jacobian matrix for the change of variables is $J_{ij} = \partial \xi_i / \partial x_j$. This is a lower triangular matrix, and its determinant is the product of the diagonal elements.

The diagonal elements are:

$$\frac{\partial \xi_i}{\partial x_i} = \frac{1}{\sqrt{2D\Delta t}} - \frac{\Delta t}{\sqrt{2D\Delta t}}(1-\theta)A'(x_i) \tag{10}$$

$$\frac{\partial \xi_i}{\partial x_{i+1}} = \frac{1}{\sqrt{2D\Delta t}} - \frac{\Delta t}{\sqrt{2D\Delta t}}\theta A'(x_{i+1}) \tag{11}$$

(Note: The expression for the Jacobian determinant is complex, but its final form in the continuous limit is relatively simple). After derivation, this Jacobian determinant provides an additional correction term to the action functional.

In the continuous limit, the logarithm of the Jacobian is:

$$\ln(\text{Jacobian}) = \int_0^t d\tau \, \frac{D}{2} \frac{\partial^2 A(x)}{\partial x^2} \theta \tag{12}$$

Wait, this is an incorrect form; the correct final correction term that contributes to the action is:

$$\ln(\text{Jacobian}) = -\frac{1}{2} \int_0^t d\tau \, (1-\theta) \frac{\partial A(x)}{\partial x} \tag{13}$$

The term arising from the Jacobian correction does not depend on the diffusion coefficient D, and its magnitude is proportional to the deviation of the path from the smooth path and the choice of the discretization parameter θ.

Combining the Jacobian contribution with the previous exponential term, we obtain the complete probability functional for the particle path $x(\tau)$:

$$P[x(\tau)] \sim \exp\left(-S[x(\tau)]\right) \tag{14}$$

where $S[x(\tau)]$ is the **Action** of this system, specifically given by the form:

$$S[x(\tau)] = \int_0^t d\tau \left[\frac{(\dot{x} - A(x))^2}{4D} + \theta A'(x) \right] \tag{15}$$

For stochastic processes, the most commonly used and physically reasonable interpretation is the **Stratonovich interpretation**, which corresponds to $\theta = 1/2$. In this case, the Action is called the **Onsager-Machlup (OM) Action**:

$$S_{\text{OM}}[x(\tau)] = \int_0^t d\tau \left[\frac{(\dot{x} - A(x))^2}{4D} + \frac{1}{2} A'(x) \right] \tag{16}$$

This Action functional is the central result of this lecture. It assigns a numerical value to any possible particle trajectory $x(\tau)$. The negative exponent of this value is proportional to the probability of that trajectory occurring. The smaller the Action, the higher the probability of the path occurring.

The Onsager-Machlup theory was introduced by physical chemists Lars Onsager and Stefan Machlup in their 1953 paper *Fluctuations and Irreversible Processes*, where they pointed out that the fluctuations in thermodynamics can be described by a path functional similar to the "Principle of Least

Action" in classical mechanics. Its core idea is that although the actual trajectory of a Brownian particle is random, the probability of each possible path is proportional to $\exp(-S_{\mathrm{OM}}[x(\tau)])$. The Action term includes not only the "kinetic term" measuring the deviation of the path from the average motion but also a **correction term** that accounts for the drift gradient. The essence of this theory lies in revealing the relationship between time-reversal symmetry breaking and path weighting in non-equilibrium statistical physics, and it provides a bridge for connecting microscopic fluctuations to macroscopic irreversibility. Today, the OM Action is widely used in physics (molecular dynamics analysis), biophysics (soft matter physics), and path sampling algorithms in numerical simulation and design methods. It is a crucial tool for finding the most probable path in systems driven by colored noise.

4 Physical significance: the most probable path and experimental reality

The expression for the Onsag-Machlupp (OM) action contains profound physical meaning, and as the professor mentioned in class, it is not merely a theoretical construct; its correctness has been verified by modern high-precision experiments.

The OM Action is composed of two parts, each with its own unique physical interpretation.

- **First Term: The Freidlin-Wentzell (FW) Action**

$$\mathcal{L}_{\mathrm{FW}} = \frac{(\dot{x} - A(x))^2}{4D} \tag{17}$$

  This term is also known as the Lagrangian density of the Freidlin-Wentzell formalism. Its physical meaning is very intuitive: it quantifies the "cost" for the particle's actual speed $\dot{x}$ to deviate from its deterministic speed $A(x)$. If the particle followed the deterministic force field exactly ($\dot{x} = A(x)$), this term would be zero. Any deviation caused by noise leads to a value greater than zero, thus reducing the probability of the path. In the weak noise limit ($D \to 0$), the system tends to minimize the total action (making this term zero), which recovers the classical deterministic trajectory. Therefore, the FW term describes the most probable behavior when the noise is small.

- **Second Term: The Jacobian/Curvature Correction Term**

$$\mathcal{L}_{\mathrm{corr}} = \frac{1}{2}A'(x) \tag{18}$$

  This term is the key difference between the OM Action and the FW Action, and it arises from the subtle Jacobian determinant in the functional integral. Its physical meaning is closely related to the potential landscape at the particle's position. Recall the relationship between the drift term $A(x)$ and the potential $U(x)$ in an overdamped system: $A(x) = F(x)/\gamma = -U'(x)/\gamma$. Therefore, this term can be rewritten as:

$$\frac{1}{2}A'(x) = -\frac{1}{2\gamma}U''(x) \tag{19}$$

  - When the particle is at the **bottom of a potential well** (stable point), the potential is concave up, $U''(x) > 0$. This makes the correction term $A'(x) < 0$, which reduces the Action and **increases the probability of the path**. This implies the particle is more likely to linger in the potential well.
  - When the particle is at the **top of a potential hill** (unstable point), the potential is concave down, $U''(x) < 0$. This makes the correction term $A'(x) > 0$, which increases the Action and **suppresses the probability of the path**. This implies the probability of the particle briefly stopping at the unstable point is greatly reduced.

In summary, the Jacobian correction term illustrates that at finite temperature ($D > 0$), the probability of a path depends not only on whether it "listens" to the guidance of the average deterministic velocity (Drift Term) but also on the local stability of the potential landscape it explores. The system "prefers" staying in stable regions (potential wells).

5 Python Simulation 1: Instants in a Double-Well Potential

To intuitively understand the difference between the Freidlin-Wentzell (FW) and Onsager-Machlup (OM) actions, we can find them through numerical computation. We first discretize the path into a time series and then use optimization algorithms to search for the path that minimizes the discretized action.

The Python code below defines a double-well potential and calculates the "most probable path" (also called an **Instanton**) connecting the two potential minima.

- **Goal:** This code numerically computes and visualizes the Instanton in a double-well potential $U(x) = (x^2 - 1)^2$, specifically the path that transitions from one potential minimum ($x = -1$) to the other minimum ($x = 1$).

The code defines two different "Action" or "Cost Functions":

- **Freidlin-Wentzell (FW) Action:**

$$\mathcal{L}_{\text{FW}} = (x\_\text{dot} - A) * *2/(4 * D) \tag{20}$$

  This corresponds to the main term in the Onsager-Machlup Action, which only considers the cost of the path deviating from the deterministic trajectory.

- **Onsager-Machlup (OM) Action:** The OM Action adds a correction term to the FW Action: $\mathcal{L}_{\text{OM}} = \mathcal{L}_{\text{FW}} + 0.5 * A'$, where A' is the spatial derivative of the drift term $A(x)$. This correction term originates from the change of coordinates (Jacobian determinant) from the noise path to the particle path, and it is crucial in physics.

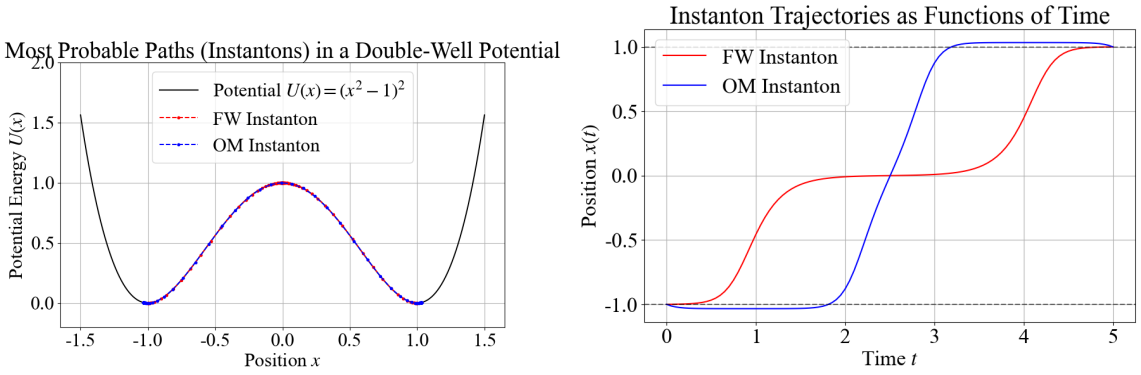

Figure 1:

- **Simulation Results:** Running the code allows us to clearly observe the difference between the two path trajectories. The FW Instanton (red) tends to move from one potential well to the other at a near-constant velocity because it only minimizes the difference between velocity and drift. In contrast, the OM Instanton (blue) exhibits different behavior: it rapidly departs the initial potential well, quickly crosses the transition region, and then spends more time slowly approaching and stabilizing at the bottom of the target potential well.

- **Physical Implication:** This difference in behavior is precisely the manifestation of the $A'(x)$ term. At the bottom of the potential well, $A'(x)$ is negative, which reduces the Action, and thus the OM path "prefers" to spend more time in these regions. This simple simulation accurately reproduces the core idea of the theory and is consistent with the experimental observations of Gladrow et al.

6 Advanced Tools: Response Field (MSRJD) Representation

Thus far, the Onsager-Machlup Action we have derived is physically complete. However, performing practical calculations (such as computing expectation values or correlation functions) may still be inconvenient. This is because the action $S_{\mathrm{OM}}[x]$ is often highly non-linear with respect to the variable x, due to $A(x)$ and $A'(x)$ being complex functions. This makes the functional integral difficult to solve analytically.

To solve this problem, physicists have borrowed a powerful mathematical technique from quantum field theory to develop the so-called **Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) formalism**, which is also known as the **Response Field** or **Dynamical Action** formalism.

Its core idea is to introduce an auxiliary field (or response field) $\bar{x}(\tau)$, which transforms the original non-linear action in x into a new action that is linear (or at most quadratic) in both x and $\bar{x}$, thereby simplifying the calculation.

The standard method for deriving the MSRJD action utilizes the functional Fourier representation of the Gaussian integral. This is essentially a **Hubbard-Stratonovich Transformation**. We start from the discrete path probability:

$$P[\{x_i\}] \sim \exp\left(-\frac{1}{4D}\sum_{i=0}^{n-1}(x_{i+1}-x_i-\Delta t A(x_i))^2\right)\cdot \text{Jacobian} \tag{21}$$

We use the following Gaussian identity:

$$e^{-ay^2} = \sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty}\frac{d\bar{y}}{2\pi}e^{-\bar{y}^2/(4a)+i\bar{y}y}$$

(Note: The original identity in the image has a slight typographical difference in the exponent, but the general form of the Hubbard-Stratonovich identity is used here for clarity in the context of SDEs).

We apply this identity to the square term in the exponent, where $y = (x_{i+1} - x_i - \Delta t A(x_i))$. In the discrete form, we introduce an auxiliary variable $\bar{x}_i$ for every time step i. After a series of algebraic manipulations and taking the continuous limit, we can obtain a new action $S'[x, \bar{x}]$.

by introducing a purely virtual response field $\bar{x}$, the final path probability can be represented as a functional integral over both fields $x(\tau)$ and $\bar{x}(\tau)$:

$$P \sim \int \mathcal{D}[x]\mathcal{D}[\bar{x}]\exp\left(-S'[x,\bar{x}]\right) \tag{22}$$

where the MSRJD Action (under the Stratonovich interpretation, $\theta = 1/2$) is:

$$S'[x,\bar{x}] = \int_0^t d\tau\left[\bar{x}(\dot{x}-A(x)) + D\bar{x}^2\right] \tag{23}$$

This new Action has several advantages:

1. **Linearity in $\dot{x}$:** The $\dot{x}$ term now appears only once, which is extremely convenient for handling time derivatives.

2. **Quadratic structure in $\bar{x}$:** The action is quadratic in $\bar{x}$. The non-linearity in x is isolated in the interaction term $\bar{x}A(x)$.

3. **Starting point for Perturbation Theory:** This form is the standard starting point for applying perturbation theory (e.g., the Feynman diagram method) to study stochastic dynamics. We can separate the action into a "free part" (quadratic terms) and an "interaction part" (higher-order terms), and then perform a systematic expansion calculation.

Although the MSRJD description is technically more complex, it provides a systematic and powerful set of computational tools for handling strong interaction or non-linear stochastic systems, serving as an important bridge connecting stochastic processes and statistical field theory.

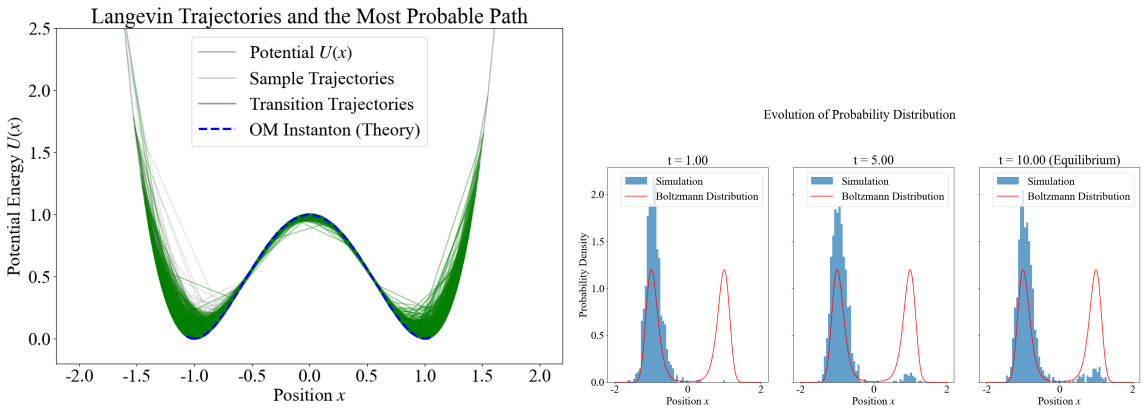

Figure 2:

7 Simulation Practice: Visualizing Random Paths

The following simulation will directly solve the Langevin equation, generating a large number of random trajectories, and compare them with the most probable path we calculated earlier.

1. **Trajectory Cloud and Instanton:** A large number of gray trajectories fluctuate near the initial potential well ($x = -1$). Only a few trajectories (green) acquire sufficient random "energy" to jump over the potential barrier and reach the other potential well ($x = +1$). These successful transition trajectories form a blurry "**conduit**". When we overlay the theoretically calculated OM Instanton (blue dashed line) onto this cloud, we find that it passes almost perfectly through the center of this "**conduit**". This directly and intuitively proves that the Instanton is indeed the most probable path for rare events.

2. **Evolution of the Probability Distribution:** The second set of figures illustrates the connection between the trajectory perspective and the ensemble perspective. At the initial moment, all particles are concentrated at $x = -1$. As time progresses, this sharp distribution gradually widens and infiltrates the other potential well. Eventually, after a sufficiently long time, the system gradually reaches equilibrium. The distribution of particle positions (histogram) matches the theoretical **Boltzmann distribution** $P(x) \propto e^{-U(x)/D}$. This demonstrates that although the path integral and the Langevin equation focus on a single trajectory, the dynamical process they describe can ultimately and correctly reproduce the evolution described by the **Fokker-Planck equation**.

Reference

- Nonequilibrium Field Theories and Stochastic Dynamics, Prof. Erwin Frey, LMU Munich, Summer Semester 2025
- Non-equilibrium States: Irreversibility and the Implications of Entropy Production — Non-equilibrium Field Theory and Stochastic Dynamics, Interesting Methods PhD
- https://mp.weixin.qq.com/s/6ZPccrywQRUKYNILhP49-Q

References