Nonequilibrium Field Theories and Stochastic Dynamics- Path integral of a system with multiplicative noise

Zehong Liao

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1 Basic description of multi-variable systems

Before going to the derive the path integral, we at first review these two equivalent mathematical frameworks, which describe the multi-variable system.

1.1 Langevin equation: Microscopic perspective of particles

The multi-variable stochastic differential equation (SDE), under Itô's interpretation, describes the evolution of the system state vector x_t over time. Its differential form is given by:

$$dx_t = A(x_t, t)\, dt + C(x_t, t)\, dW_t \tag{1}$$

where:

- x_t is a d-dimensional vector, representing the state of the system at time t (e.g., position, velocity, etc., of a particle).
- $A(x_t, t)$ is the **drift vector**, representing the deterministic forces or tendencies acting on the system. It describes the average direction of motion of the system in the absence of random fluctuations.
- dW_t is a d-dimensional infinitesimal increment of a Wiener process (or Brownian motion). Each of its components is an independent, normally distributed random variable, with a probability distribution given by:

  $$P(dW_t) = \frac{1}{(2\pi dt)^{d/2}} \exp\left(-\frac{1}{2dt}|dW_t|^2\right) \tag{2}$$

  This implies that dW_t has a mean of zero and a variance of dt.
- $C(x_t, t)$ is the **noise matrix** (or diffusion matrix). It describes how the random forces act on the different degrees of freedom of the system.

The focus of this lecture section is on **state-dependent noise** (multiplicative noise), where its mathematical characteristic lies in the **noise matrix** C depending on the current state of the system x_t. This has profound physical implications: **the strength or characteristics of the random perturbation depend on the location of the system in the state space.** For example, in financial models, the amplitude of stock price fluctuations is often proportional to the price itself; in biological population models, environmental random fluctuations have a greater impact on larger populations. This state-dependent noise makes the dynamics of the system far more complex and rich than additive noise (where C is a constant).

1.2 Fock-Planck equations: A macroscopic perspective of an ensemble

Unlike the single trajectory described by the Itô SDE, the Fokker-Planck equation describes how the probability density function $p(\mathbf{x}, t)$ of a system composed of a large number of identical systems evolves over time:

$$\partial_t p(\mathbf{x}, t) = -\sum_i \partial_i [A_i(\mathbf{x}, t) p(\mathbf{x}, t)] + \frac{1}{2} \sum_{i,j} \partial_i \partial_j [B_{ij}(\mathbf{x}, t) p(\mathbf{x}, t)] \tag{3}$$

This equation can be understood as a conservation equation for probability in state space.

- The first term is the **drift term**, which describes how the "center" or "peak" of the probability distribution moves following the deterministic drift field A.
- The second term is the **diffusion term**, which describes how the probability distribution gradually broadens or diffuses as time progresses.

There is a crucial relationship between these two descriptions: the relationship between the **diffusion tensor** B and the **noise matrix** C:

$$B(\mathbf{x}, t) = C(\mathbf{x}, t) \cdot C^T(\mathbf{x}, t) \tag{4}$$

This relationship precisely encodes how the statistical characteristics of the microscopic noise are transformed into the macroscopic diffusion behavior of the probability density. The matrix product $C \cdot C^T$ essentially computes the **covariance matrix of the noise** transformed by C. If C is a diagonal matrix, it means the noise in different directions is statistically independent; if C contains off-diagonal elements, then there is a correlation between the random forces in different directions. The diffusion tensor B perfectly captures these correlations and determines the shape in which the probability cloud diffuses in multi-dimensional space (e.g., circular or elliptical). Thus, this formula is the key link between the microscopic (Itô) and macroscopic (Fokker-Planck) descriptions.

2 Derivation of the path integral of stochastic differential equations

Now, we will start from the Langevin equation, and construct an equivalent path integral representation step by step.

We cannot directly handle continuous stochastic differentials, so the first step is to discretize them. We divide the total time T into N small time steps $\Delta t = T/N$. For any time step, from t_i to t_{i+1}, the Itô equation can be approximately written as:

$$x_{i+1} - x_i = A(x_i)\Delta t + C(x_i)\Delta W_i \tag{5}$$

Here, $\Delta W_i = W(t_{i+1}) - W(t_i)$ is a Gaussian random vector with a mean of 0 and a covariance matrix of $I\Delta t$. This discretization method is called the **Euler-Maruyama method**. The Euler-Maruyama (EM) method is a classic numerical algorithm for solving stochastic differential equations (SDEs), developed from the 18th-century Euler method for ordinary differential equations (ODEs). The Euler method relies on computing numerical derivatives by utilizing the time step and forward difference approximations. In the 20th century, Japanese mathematician Gisiro Maruyama extended it to the field of stochastic differential equations, introducing the Euler-Maruyama method. By incorporating the Wiener process (Brownian motion increment) to model system evolution, the physical essence of this method is the combination of deterministic dynamics and random perturbations, using an iterative formula to approximate the solution of the SDE, making it suitable for describing dynamically evolving systems affected by noise. It is widely applied, including option pricing in finance (like the Black-Scholes model), molecular dynamics simulations in physics, kinetic modeling in biomechanics, and sampling in machine learning models (like image generation). Although the EM method has the advantages of being computationally simple and easy to implement, its convergence is relatively low and it is sensitive to step size, typically requiring high precision or systematic initial analysis.

A crucial detail is that the drift term A and the noise matrix C are sampled at the **start point** of the time step, x_i. This **non-anticipating** sampling method is the core characteristic of the Itô integral. It is the choice made at the time of this derivation that determines the solution we eventually obtain corresponds to the Itô interpretation. If we were to choose the midpoint of the time step for sampling, it would lead to another type of stochastic integral—the Stratonovich integral—the difference between which will be discussed at the end of the course.

Now, let's calculate the probability of the system transitioning from state x_i to x_{i+1} within a single time step, which is the short-time propagator $P(x_{i+1}|x_i)$.

From the discretized Itô SDE, since ΔW_i is a Gaussian random variable, the displacement $x_{i+1} - x_i$ is also Gaussian-distributed. We can compute its mean and covariance:

- **Mean:**
$$\mu_i = x_i + A(x_i)\Delta t \tag{6}$$

- **Covariance Matrix:**
$$E[(C(x_i)\Delta W_i)(C(x_i)\Delta W_i)^T] = C(x_i)E[\Delta W_i \Delta W_i^T]C(x_i)^T = C(x_i)(I\Delta t)C(x_i)^T = \Delta t\, C(x_i)C(x_i)^T = \Delta t\, B(x_i) \tag{7}$$

Therefore, the single-step transition probability is a multivariate Gaussian distribution:

$$P(x_{i+1}|x_i) = \frac{1}{\sqrt{(2\pi\Delta t)^d \det B(\vec{x}_i)}} \exp\left[-\frac{1}{2\Delta t}(\vec{x}_{i+1} - \vec{\mu}_i)^T B^{-1}(\vec{x}_i)(\vec{x}_{i+1} - \vec{\mu}_i)\right] \tag{8}$$

This expression is mathematically precise, but the matrix inverse B^{-1} appearing in the exponential term will cause great trouble in subsequent calculations. Our next step is to use mathematical techniques to eliminate it.

To handle the matrix inverse in the exponential term, we employ a very powerful mathematical tool: the Fourier representation of a Gaussian integral. Any Gaussian function can be expressed as the differential form of its Fourier transform. Using the following Gaussian integral identity:

$$\exp\left[-\frac{1}{2}\vec{v}^T \mathbf{M}^{-1}\vec{v}\right] = \frac{\sqrt{\det \mathbf{M}}}{\sqrt{(2\pi)^d}} \int d^d\vec{q} \exp\left[-\frac{1}{2}\vec{q}^T \mathbf{M}\vec{q} + i\vec{q}^T\vec{v}\right] \tag{9}$$

Applying this identity to our short-time propagator, where we set $\vec{v} = \vec{x}_{i+1} - \vec{\mu}_i$ and $\mathbf{M} = B(\vec{x}_i)\Delta t$, we can rewrite $P(\vec{x}_{i+1}|\vec{x}_i)$ as an integral over an auxiliary variable $\vec{q}$:

$$P(\vec{x}_{i+1}|\vec{x}_i) = \int \frac{d^d\vec{q}}{(2\pi)^d} \exp\left[-\frac{1}{2}\Delta t\, \vec{q}_{i+1}^T B(\vec{x}_i)\vec{q}_{i+1} + i\vec{q}_{i+1}^T(\vec{x}_{i+1} - \vec{x}_i - A(\vec{x}_i)\Delta t)\right] \tag{10}$$

This transformation is the key step in the entire derivation. We have introduced a new integration variable $\vec{q}$, which is physically called the **response field**. Through this transformation, we have successfully replaced the complex matrix inverse B^{-1} in the original exponential term with a simple quadratic term $\vec{q}^T B\vec{q}$. This paves the way for us to chain many time steps together.

A complete path is the evolution process from the initial state x_0 at time t_0, through a series of intermediate states $x_1, \dots, x_{N-1}$, finally reaching the final state x_N at time t_N. The total transition probability can be obtained by multiplying all the single-step transition probabilities together and integrating over all possible intermediate states, based on the Markov property (the **Chapman-Kolmogorov equation**):

$$P(x_N, t_N|x_0, t_0) = \int dx_1 \cdots \int dx_{N-1} \prod_{i=0}^{N-1} P(x_{i+1}|x_i) \tag{11}$$

Now, we substitute the Fourier representation for the single-step propagator $P(x_{i+1}|x_i)$ for every step:

$$P(\vec{x}_f, t_f|\vec{x}_0, t_0) = \left(\prod_{i=1}^{N-1} d\vec{x}_i\right)\left(\prod_{j=0}^{N-1} \frac{d\vec{q}_j}{(2\pi)^d}\right) \exp\left[-\sum_{k=1}^{N} \frac{1}{2}\Delta t \vec{q}_{k-1}^T B(\vec{x}_{k-1})\vec{q}_{k-1} + i\sum_{k=1}^{N} \vec{q}_{k-1}^T(\vec{x}_k - \vec{x}_{k-1} - A(\vec{x}_{k-1})\Delta t)\right] \tag{12}$$

Here, $x_f = x_N$. This expression appears complex, but it already has the form of a **Path Integral**. We are integrating over all intermediate positions $\vec{x}_i$ and all auxiliary fields $\vec{q}_j$.

The last step is to take the continuous limit, where the number of time steps $N \to \infty$ and the time step $\Delta t \to 0$. In this limit:

- The discrete sum $\sum_{k=1}^{N} \Delta t(\dots)$ transforms into a time integral $\int_{t_0}^{t_f} dt(\dots)$.
- The finite difference $(x_k - x_{k-1})/\Delta t$ becomes the time derivative $\partial_t x$.
- The infinite product integral over all discrete variables x_i and q_j is formally written as a **functional integral** or **path integral**, denoted as $\mathcal{D}[x]\mathcal{D}[q]$.

After reorganization, we obtain the final path integral expression:

$$P(x_f, t_f|x_0, t_0) = \int_{x(t_0)=x_0}^{x(t_f)=x_f} \mathcal{D}[x]\mathcal{D}[q] \exp\left(-S[x, iq]\right) \tag{13}$$

where the exponent S is called the **action**, and its explicit form is:

$$S[\vec{x}, i\vec{q}] = \int_{t_0}^{t_f} dt \left[i\vec{q}^T(\partial_t \vec{x} - \vec{A}(\vec{x})) + \frac{1}{2}\vec{q}^T \mathbf{B}(\vec{x})\vec{q} \right] \tag{14}$$

This action is known in non-equilibrium statistical physics as the **Martin-Siggia-Rose-Janssen-De Dominicis (MSRJD) action**, which we will briefly introduce in Chapter 18. It beautifully re-expresses a stochastic differential equation problem as a field theory problem.

3 Physical interpretation of the action of the path integral

The first term in the action is $i\vec{q}^T(\partial_t\vec{x}-\vec{A}(\vec{x}))$. In the functional integral, integrating over the response field $\vec{q}(t)$ across all phase spaces, which acts like a Lagrange multiplier, yields a **functional delta function**:

$$\int \mathcal{D}[q] \exp\left(\int dt\, i\vec{q}^T(\partial_t \vec{x} - \vec{A}(\vec{x})) \right) \propto \delta[\partial_t \vec{x} - \vec{A}(\vec{x})] \tag{15}$$

This delta function is a "hard constraint" that enforces that all paths $\vec{x}(t)$ summed over in the path integral must satisfy the deterministic part of the dynamical equation, $\partial_t\vec{x} = \vec{A}(\vec{x})$ (when there is no noise). Therefore, the role of this term is to incorporate the deterministic evolution rules of the system into the path integral framework.

The second term in the action is $\frac{1}{2}(\vec{iq})^T\mathbf{B}(\vec{x})(\vec{iq})$. This is a quadratic term of the response field $i\vec{q}$. Its form is exactly the same as the exponential part of a Gaussian distribution with zero mean and covariance matrix $\mathbf{B}(\vec{x})$.

Therefore, its physical meaning is to encode all the statistical information of the noise it senses—its variance and the correlation between its different components—in the diffusion tensor $\mathbf{B}(\vec{x})$. The response field q gets its name because, in higher-order applications, it can be added to the action combined with a source term, allowing one to compute the linear response function of the system to external small perturbations.

Integrating over the stochastic variable x actually transforms the problem into a $(0+1)$-dimensional field theory (no spatial dimension, only time). In this theory, the system's state $\vec{x}(t)$ and the response field $\vec{q}(t)$ act as **dynamical "field variables"**. This perspective is extremely powerful because it allows us to "transplant" powerful tools developed in quantum field theory (like Feynman diagrams, perturbation theory, and the renormalization group) to analyze classical stochastic processes, which is particularly effective for studying phase transitions, critical phenomena, and complex dynamics.

In addition to the strict derivation method presented above, we can reach the same action using a more physically intuitive and heuristic approach. This method helps us better understand the origin and physical meaning of the response field q.

1. **Starting from the Noise Source:** Write the Itô equation in a form that includes a standard white noise source $\eta(t)$:

$$\partial_t \mathbf{x} = \mathbf{A}(\mathbf{x}) + \mathbf{C}(\mathbf{x})\boldsymbol{\eta}(t)$$

   where the probability of any noise path $\boldsymbol{\eta}(t)$ is determined by a simple weighted exponential:

$$P[\boldsymbol{\eta}] \sim \exp\left(-\frac{1}{2}\int dt\, \boldsymbol{\eta}^T(t)\boldsymbol{\eta}(t)\right)$$

2. **Expectation Value Calculation:** The expectation value of any observable $\mathcal{O}(\mathbf{x})$ can be calculated by first solving the trajectory $\mathbf{x}(t)$ for a fixed noise path $\boldsymbol{\eta}(t)$, and then performing a weighted average over all possible noise paths:

$$\langle \mathcal{O}(\mathbf{x}) \rangle = \int \mathcal{D}[\boldsymbol{\eta}]\mathcal{O}(\mathbf{x}[\boldsymbol{\eta}])P[\boldsymbol{\eta}]$$

3. **Imposing the Constraint via a Functional Delta Function:** To couple $\mathbf{x}$ and $\boldsymbol{\eta}$ within a single integral, we use a functional delta function to enforce the constraint that $\mathbf{x}(t)$ must satisfy the equation of motion $\partial_t \mathbf{x} - \mathbf{A}(\mathbf{x}) - \mathbf{C}(\mathbf{x})\boldsymbol{\eta}(t) = 0$:

$$\langle \mathcal{O}(\mathbf{x}) \rangle = \int \mathcal{D}[\mathbf{x}]\mathcal{D}[\boldsymbol{\eta}]\mathcal{O}(\mathbf{x})\, \delta[\partial_t \mathbf{x} - \mathbf{A}(\mathbf{x}) - \mathbf{C}(\mathbf{x})\boldsymbol{\eta}] \exp\left(-\frac{1}{2}\int dt\, \boldsymbol{\eta}^T \boldsymbol{\eta}\right)$$

4. **Delta Function Fourier Representation:** Next, we write the delta function in terms of its Fourier integral representation. This step naturally introduces the response field q:

$$\delta[\dots] = \int \mathcal{D}[q] \exp\left(i \int dt\, \mathbf{q}^T (\partial_t \mathbf{x} - \mathbf{A}(\mathbf{x}) - \mathbf{C}(\mathbf{x})\boldsymbol{\eta})\right)$$

5. **Integrating over the Physical Noise $\boldsymbol{\eta}$:** Substituting this back, the expression now involves integrals over $\mathbf{x}$, $\mathbf{q}$, and the physical noise $\boldsymbol{\eta}$. We can first perform the integral over the physical noise $\boldsymbol{\eta}(t)$. The integral over $\boldsymbol{\eta}$ is a standard Gaussian integral because $\boldsymbol{\eta}$ appears only in a quadratic term (from $P[\boldsymbol{\eta}]$) and a linear term (from the coupling with $\mathbf{q}$). Completing the square to perform this Gaussian integral, we get:

$$\int \mathcal{D}[\boldsymbol{\eta}] \exp\left(-\frac{1}{2}\int dt\, \boldsymbol{\eta}^T \boldsymbol{\eta} + i \int dt\, \mathbf{q}^T \mathbf{C}(\mathbf{x})\boldsymbol{\eta}\right) = \exp\left(-\frac{1}{2}\int dt\, i\mathbf{q}^T \mathbf{C}(\mathbf{x})\mathbf{C}^T(\mathbf{x}) i\mathbf{q}\right)$$

Combining this result with the remaining terms in the exponent, we find that the action depends only on the fields $\mathbf{x}$ and $\mathbf{q}$, and is exactly identical to the result obtained from our rigorous derivation.

This heuristic derivation clearly reveals the physical essence of the response field $\mathbf{q}$: it is the field conjugate (in the Fourier sense) to the dynamic constraint imposed by the equation of motion. The term containing $\mathbf{q}$ is derived from the coupling between the dynamic constraint and the physical noise $\boldsymbol{\eta}$. In this sense, the $\mathbf{q}$ part of the action can be considered the "trace" left behind after integrating out the physical noise.

4 Simulation and Visualization: Geometric Brownian Motion in Python

The stochastic differential equation (SDE) for Geometric Brownian Motion is as follows:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{16}$$

where:

- μ is the drift rate.
- σ is the volatility rate.

This is a perfect example of multiplicative noise, because both the deterministic drift term $\mu X_t dt$ and the stochastic diffusion term $\sigma X_t dW_t$ are proportional to the current state X_t.

This model is widely used in many fields, for example, in financial mathematics, where it is used to model stock prices (the higher the price, the larger the absolute value of the fluctuation, and the price will not be negative). In biology, it can describe population growth under unlimited resources, where both the growth and fluctuations are proportional to the current population size.

We will use the previously discussed Euler-Maruyama method to implement a numerical simulation of GBM. Its discrete form is:

$$X_{n+1} = X_n + \mu X_n \Delta t + \sigma X_n \Delta W_n = X_n(1 + \mu\Delta t + \sigma\sqrt{\Delta t} Z_n) \tag{17}$$

where Z_n is a random number drawn from the standard normal distribution (mean 0, variance 1).

The most noticeable characteristic of 50 GBM paths starting from the same point is that all trajectories exhibit a **"fanning out"** pattern. This is the most direct manifestation of multiplicative noise.

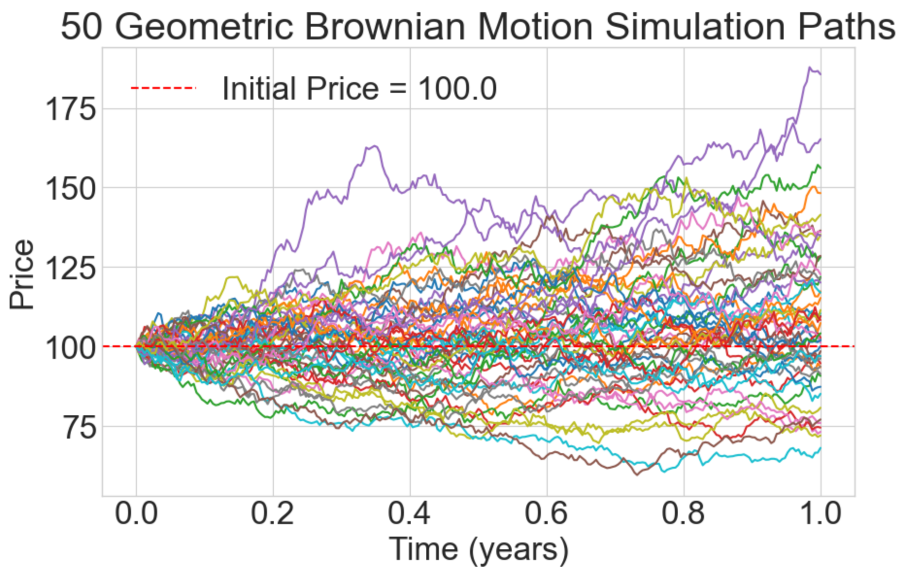

Figure 1:

- When a trajectory reaches a higher value due to a random fluctuation, its state X_t increases.
- Because the noise term $\sigma X_t dW_t$ is proportional to X_t, a larger X_t leads to a larger amplitude of the random fluctuation.
- This, in turn, makes the trajectory more likely to undergo dramatic upward jumps, further increasing the distance between it and other trajectories.

In contrast, for additive noise (such as standard Brownian motion), the noise intensity is constant, and the degree of diffusion for all trajectories is statistically uniform, forming a symmetric, spreading distribution. However, the multiplicative noise of Geometric Brownian Motion leads to an asymmetric distribution that becomes wider over time and is skewed towards the right (a long tail).

5 Summary

This lesson systematically establishes the path integral formulation of stochastic processes with multiplicative noise. Starting from the discretized Langevin equations, we derive the action of the MSRJD and explore the physical meaning of its terms in depth. The path integral not only provides a completely new theoretical perspective but also connects stochastic processes with statistical field theory, offering a powerful computational tool for solving complex problems.

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